The thermodynamics of site-random mean-field quantum spin systems

1989 J. Phys. A: Math. Gen. 22 4643

(http://iopscience.iop.org/0305-4470/22/21/025)

J. Phys. A: Math. Gen. 22 (1989) 4643-4658. Printed in the UK

The thermodynamics of site-random mean-field quantum spin systems

N G Duffield†‡ and R Kühn§

Universität Heidelberg, Sonderforschungsbereich 123, Im Neuenheimer Feld 294, D-6900 Heidelberg, Federal Republic of Germany

Received 30 March 1989

Abstract. We provide a general scheme for the treatment of the thermodynamics of mean-field site-random quantum spin systems, including systems where bond randomness is expressed as an underlying site randomness. We use the method to find the phase structure of a mean-field Heisenberg model in a random field, and of a mean-field Heisenberg spin glass model.

1. Introduction

Considerable interest has been devoted in recent years towards the study of randomly disordered spin systems, such as spin glasses [1], dilute magnets [2], or systems in random fields [3]. While there are a number of rigorous results available (see e.g. [4]), exact solutions of disordered model systems are almost impossible to obtain if interactions are of short range and the dimension of the underlying lattice exceeds 1, the model of McCoy and Wu [5] being a notable exception. Some progress has, however, been made for systems with infinite-range interactions where the mean-field approximation becomes exact [6-8]. But even here, the true random bond problems have posed considerable difficulties, their solution requiring recourse to the $n \to 0$ replica trick and sophisticated schemes of (hierarchical) replica symmetry breaking [9].

Fairly general methods of solution do, however, exist for random site models, including those where bond randomness is expressed in terms of an underlying site randomness [7-11]. Models of this type have been proposed to describe the thermodynamics of spin glasses, but are currently also widely studied in the context of formal neural networks.

It then seems natural to ask—though perhaps not with an eye on neural network models—whether solutions of similar generality can be obtained if one replaces classical spins by quantum spins. It is the purpose of the present paper to demonstrate that for general site-random Curie-Weiss models described by Hamiltonians of the form

$$\tilde{H}_N(\boldsymbol{\xi}) = -\sum_{i=1}^{N} \sum_{\mu=x,y,z} \sigma_i^\mu Q_1^\mu(\xi_i) - \frac{1}{2N} \sum_{i,j=1}^{N} \sum_{\mu,\mu'=x,y,z} \sigma_i^\mu Q_2^{\mu\mu'}(\xi_i;\xi_j)\sigma_j^{\mu'} \tag{1.1}$$

the answer is in the affirmative. Here

$$\sigma_i^\mu = 1 \otimes \ldots \otimes 1 \otimes \sigma^\mu \otimes 1 \otimes \ldots \otimes 1 \in (\mathbb{C}^2)^{\times N} \tag{1.2}$$

† Supported by a European Science Exchange Fellowship of the Royal Society, London.
‡ Present address: Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland.
§ Supported by the Deutsche Forschungsgemeinschaft.

0305-4470/89/214643+16$02.50

where σ^μ: $\mu = x, y, z$ is a Pauli spin matrix, i denotes the ith position in the tensor product, and $\boldsymbol{\xi}$ is a collection $\{\xi_i\}$ of random vectors in $\mathbb{R}^q$. We will obtain a variational expression for the free energy for such quadratic random Hamiltonians. However, the methods can be applied straightforwardly to any polynomial Hamiltonian of this random type.

The basic ideas involved are most easily explained if the distribution of the ξ_i is discrete. In that case, the Hamiltonian (1.1) only depends on the *total spin operators* over the sublattices (or subsets) of $\{1, \ldots, N\}$ on which the values of the ξ_i coincide [10, 11]. To deal with the quantum nature of the spins, two further ingredients are essential: the decomposition of total spin operators into a sum of irreducible spin operators, and the bounding of quantum partition functions by classical partition functions using the Berezin–Lieb inequalities [12–14].

Any total spin operator $S_N^\mu = \frac{1}{2}\Sigma_{i=1}^N \sigma_i^\mu$ can be decomposed into irreducible representations of SU(2):

$$S_N^\mu = \bigoplus_{J=0 \text{ or } \frac{1}{2}}^{N/2} \bigoplus_{k=1}^{c(N,J)} {}^J S^{\mu,k}$$

where J runs over the integers (half-integers) if N is even (odd), ${}^J S^{\mu,k}$ is a copy of ${}^J S^\mu$, the representation of the μ spin component in the $(2J+1)$-dimensional representation of SU(2), and the $c(N, J)$ are the multiplicities of the decomposition. It is found [15, 16] that the $c(N, J)$ have the asymptotic form

$$c(N, J) \sim \exp(-NI(r)) \tag{1.3}$$

where I is some smooth function of $r = 2J/N$ (see below).

The Berezin–Lieb inequalities state, in particular, that for any operator ${}^J H$ polynomial in the ${}^J S^\mu$ there exist functions $({}^J H)^u$ and $({}^J H)^l$, called the upper and lower symbols of ${}^J H$, defined on the sphere S^3, such that

$$\frac{(2J+1)}{4\pi}\int_{S^3} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{e}^{-\beta({}^J H)^l(\theta,\phi)} \leqslant \operatorname{tr} \mathrm{e}^{-\beta {}^J H} \leqslant \frac{(2J+1)}{4\pi}\int_{S^3} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{e}^{-\beta({}^J H)^u(\theta,\phi)}. \tag{1.4}$$

If we have a Hamiltonian of the form N times a polynomial in S_N^μ/N, and decompose it according to the irreducible representation of SU(2), then it is found [16] that the upper and lower symbols of ${}^J H$ are of the asymptotic form

$$Nh(r, \theta, \phi) + \mathrm{O}(1)$$

for some function h. Combining (1.3) and (1.4) we see that

$$\operatorname{tr} \mathrm{e}^{H_N} \sim \int_{[0,1]\times S^3} \mathrm{d}r \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \frac{(Nr+1)}{4\pi} \mathrm{e}^{-N(I(r)+\beta h(r,\theta,\phi))} \tag{1.5}$$

and so as $N \to \infty$ we expect (1.5) to be dominated by its largest contribution, which occurs when $I + h$ is at a minimum. These ideas are made rigorous in [16], by use of the theory of large deviations [17, 18].

The Hamiltonians of the site-random models considered here depend not only on one, but on *several* total spin operators, one for each of the sublattices of constant ξ. So an extension of the above ideas to the multiple decomposition of several total spin operators is needed. This can be found in recent work of Duffield and Pulé [19, 20].

In the case of continuous probability distributions of the ξ, the sublattice techniques cannot be applied immediately. However, approximating the continuous distributions by a sequence of discrete ones, we find that the corresponding sequence of free energies converges to the desired continuum quantity. In fact, if one considers only random variables taking values in a compact set, then this approximation follows immediately from the results of [19]. However, we do not make this restriction. A corresponding proof for classical random-site models can be found in [11].

The outline of the present paper is as follows. In § 2 we consider the case of discrete random variables in a little more technical detail, and obtain a variational expression for the free energy. In § 3 we extend the method to the case of continuously distributed random variables. To illustrate the working of our ideas, we solve the variational problem for two examples, a quantum XYZ model in a random field (in § 4), and a quantum XYZ spin glass with randomly preferred interaction directions (in § 5). We compare the behaviour of these quantum models with various classical analogues.

2. Thermodynamic limit for discrete random variables

We first treat the case where the underlying random variables have a discrete distribution. We will provide a general treatment for quadratic random Hamiltonians which can then be applied to individual examples. We emphasise, however, that the results (and their proofs) carry over for arbitrary polynomial site-random mean-field Hamiltonians.

We define the random Hamiltonians as follows. Let $\Gamma = (\mathbb{R})^q$ for some integer q, and let ρ be a discrete measure on Γ:

$$\rho = \sum_{m=1}^{m(n)} p_m \delta_{\xi^m}$$

with

$$0 \leqslant p_m \leqslant 1 \qquad \text{and} \qquad \sum_{m=1}^{m(n)} p_m = 1$$

for some integer $m(n)$ where δ_ξ is the Dirac measure with support ξ. Let Q_1 be a function from Γ to $\mathbb{R}^3$, and let $Q_2 : \Gamma \times \Gamma \to M_3$ (the 3×3 matrices), obey

$$Q_2(\xi;\, \eta) = Q_2(\eta,\, \xi)^*$$

where * denotes Hermitian conjugation in M_3. Q_1 and Q_2 will couple to spin matrices in the Hamiltonian: accordingly, we label the their components by the indices x, y and z. Let $\Omega = \Gamma^{\times \mathbb{N}}$, and let P_ρ denote the infinite product measure $\rho^{\otimes \mathbb{N}}$ on Ω. Our basic random variables will be the collections $\boldsymbol{\xi} = \{\xi_i\}_{i \in \mathbb{N}} \in \Omega$, and we define the random Hamiltonian for system size N as

$$\tilde{H}_N(\boldsymbol{\xi}) = -\sum_{i=1}^{N} \sum_{\mu = x,y,z} \sigma_i^\mu Q_1^\mu(\xi_i) - \frac{1}{2N} \sum_{i,j=1}^{N} \sum_{\mu,\mu' = x,y,z} \sigma_i^\mu Q_2^{\mu\mu'}(\xi_i;\, \xi_j) \sigma_j^{\mu'} . \tag{2.1}$$

Here

$$\sigma_i^\mu = \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \sigma^\mu \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} \in (\mathbb{C}^2)^{\times N}$$

where σ^μ is a Pauli spin matrix and the index i denotes the ith position in the tensor product.

Define the random free energy

$$\tilde{f}_N(\beta, \boldsymbol{\xi}) = -\frac{1}{\beta N} \log \operatorname{tr} \exp(-\beta \tilde{H}_N(\boldsymbol{\xi})).$$

The following theorem states that this quantity is deterministic in the thermodynamic limit, and gives a variational expression for the limit.

Theorem 1. P_ρ—almost surely, $\lim_{N\to\infty} \tilde{f}_N(\beta, \boldsymbol{\xi}) = \tilde{f}(\beta)$ where

$$\tilde{f}(\beta) = -\sup_{(\boldsymbol{r},\boldsymbol{\theta},\boldsymbol{\phi})\in([0,1]\times[0,\pi]\times[0,2\pi])^{m(n)}} \left\{ \tilde{h}(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\phi}) - \frac{1}{\beta}\mathcal{I}(\boldsymbol{r}) \right\}$$

with

$$\tilde{h}_n(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\phi}) = \sum_{m=1}^{m(n)} p_m r_m \sum_\mu \mathrm{e}^\mu(\theta_m, \phi_m) Q_1^\mu(\xi^m)$$
$$+ \frac{1}{2} \sum_{m,m'=1}^{m(n)} p_m p_{m'} r_m r_{m'} \sum_{\mu,\mu'} \mathrm{e}^\mu(\theta_m, \phi_m)\, \mathrm{e}^{\mu'}(\theta_{m'}, \phi_{m'}) Q_2^{\mu\mu'}(\xi^m; \xi^{m'}) \qquad (2.2)$$

where

$$\mathrm{e}^x(\theta, \phi) = \sin\theta\cos\phi \qquad \mathrm{e}^y(\theta, \phi) = \sin\theta\sin\phi \qquad \mathrm{e}^z(\theta, \phi) = \cos\theta$$

and

$$\mathcal{I}(\boldsymbol{r}) = \sum_{m=1}^{n} p_m I(r_m)$$

where

$$I(r) = \tfrac{1}{2}(1+r)\log(1+r) + \tfrac{1}{2}(1-r)\log(1-r) - \log 2. \qquad (2.3)$$

Proof. Define the random sets

$$\Omega_m(N, \boldsymbol{\xi}) = \{i \in \{1, 2, \ldots, N\} : \xi_i = \xi^m\}$$

and define the total spin operators over these sets:

$$S_m^\mu(N, \boldsymbol{\xi}) = \frac{1}{2} \sum_{i\in\Omega_m(N,\boldsymbol{\xi})} \sigma_i^\mu .$$

Equation (2.1) can be rewritten as

$$\tilde{H}_N(\boldsymbol{\xi}) = -2 \sum_{m=1}^{m(n)} \sum_\mu S_m^\mu(N, \boldsymbol{\xi}) Q_1^\mu(\xi^m) - \frac{2}{N} \sum_{m,m'=1}^{m(n)} \sum_{\mu,\mu'} S_m^\mu(N, \boldsymbol{\xi}) Q_2^{\mu\mu'}(\xi^m; \xi^{m'}) S_{m'}^{\mu'}(N, \boldsymbol{\xi}). \qquad (2.4)$$

Now by the ergodic theorem (see e.g. [17]), $p_m(N, \boldsymbol{\xi}) = N^{-1} \# \Omega_m(N, \boldsymbol{\xi})$ converges to p_m, P_ρ-almost surely as $N \to \infty$. Furthermore, it is clear from (2.4) that $\tilde{f}_N(\beta, \boldsymbol{\xi})$ depends on $\boldsymbol{\xi}$ only through the $p_m(N, \boldsymbol{\xi})$. With these observations, the stated result is a trivial modification of theorem 2 of [19] using the table in [13] (or rather, its corrected version in [16]). A sketch of the arguments involved is given in the introduction. □

3. Continuous distributions

Given a continuous measure ρ on Γ, we can approximate it (weakly) by a sequence of discrete measures. In this section we show that the corresponding approximation for free energies derived from the discrete measures by theorem 1 is good: the continuum free energies are just the limits of the discrete free energies. First we will need the following.

Assumptions

(a) ρ is continuous and has a finite expectation.

(b) Piecewise, Q_1 is continuous (respectively Q_2 is jointly continuous), and has elements which are absolutely integrable against ρ (respectively $\rho\otimes\rho$).

We will explicitly construct the discrete approximations to ρ. For positive x, let $B(x)\subset\Gamma$ be the hypercube of side x, centred at the origin, with sides parallel to the coordinate axes. Define x_n by

$$\rho(B(x_n))=1-\frac{1}{n}.$$

Note that

$$\frac{x_n}{n}\leqslant\int_{\Gamma\setminus B(x_n)}|\xi|\,\mathrm{d}\rho(\xi)\qquad\text{where}\qquad|\xi|=\sup_{1\leqslant q'\leqslant q}|\xi^{(q')}|\tag{3.1}$$

where the $\xi^{(q')}$ are the q components of $\xi\in\Gamma$. Since ρ has a finite expectation, the RHS of (3.1) goes to zero as $n\to\infty$ and so

$$\lim_{n\to\infty}n^{-1}x_n=0.$$

Divide $B(x_n)$ into n^q disjoint hypercubes of side x_n/n. The hypercubes and $\Gamma\setminus B(x_n)$ form a partition $\mathscr{P}'_n$ of Γ. We define the partititons $\mathscr{P}_n=\bigwedge_{m=1}^{n}\mathscr{P}'_m$ of Γ, and enumerate it as $\mathscr{P}_n=\{B^n_m:m=1,2,\ldots,m(n)\}$ with $B^n_{m(n)}=\Gamma\setminus B(x_n)$. Approximate the identity map on Γ by g_n where $g_n(x)=0$ for $x\in B^n_{m(n)}$, while if $x\in B\in\mathscr{P}_n\setminus B^n_{m(n)}$ then

$$g_n(x)=\begin{cases}0 & \text{if } \rho(B)=0\\ (\rho(B))^{-1}\displaystyle\int_B \mathrm{d}\rho(x)x & \text{otherwise.}\end{cases}\tag{3.2}$$

Since x_n is non-decreasing and the spacing $n^{-1}x_n\to 0$, then clearly the step functions g_n converge pointwise to the identity function on the support of ρ in Γ.

Our discrete measures are simply $\rho_n=\rho g_n^{-1}$, i.e. those measures for which $\rho_n(A)=\rho(g_n^{-1}A)$ for A a Borel subset of Γ.

We define the discretised Hamiltonian as a function of the now *continuous* random variables $\boldsymbol{\xi}$:

$$H^n_N(\boldsymbol{\xi})=-\sum_{i=1}^{N}\sum_{\mu}\sigma_i^{\mu}Q_1^{\mu}(g_n(\xi_i))-\frac{1}{2N}\sum_{i,j=1}^{N}\sum_{\mu,\mu'}\sigma_i^{\mu}Q_2^{\mu\mu'}(g_n(\xi_i);g_n(\xi_j))\sigma_j^{\mu'}.\tag{3.3}$$

Thus H^n_N is just an approximation to the Hamiltonian H_N, the latter being obtained by replacing g_n with the identity map in (3.3).

Define the free energy $f^n_N(\beta,\boldsymbol{\xi})=-(\beta N)^{-1}\log\operatorname{tr}\exp(-\beta H^n_N(\boldsymbol{\xi}))$. Since g_n is a step function, then applying theorem 1, we see that $f^n(\beta)=\lim_{N\to\infty}f^n_N(\beta)$ is given almost surely by an expression of the form (2.2) with $p_m=\rho(B^n_m)$ and $\xi_m=g_n(B^n_m)$.

We shall write this in a slightly different way. Let $\mathcal{M}$ denote the elements (r, θ, ϕ) of $\mathcal{L}^1(\Gamma, \rho) \oplus \mathcal{L}^1(\Gamma, \rho) \oplus \mathcal{L}^1(\Gamma, \rho)$ for which $0 \leq r(x) \leq 1 : 0 \leq \theta(x) \leq 2\pi;\ 0 \leq \phi(x) \leq \pi$:

$$f^n(\beta) = -\sup_{(r,\theta,\rho)\in\mathcal{M}^n} \mathcal{S}_n(r, \theta, \phi) \tag{3.4}$$

where $\mathcal{M}^n$ is the set of step functions in $\mathcal{M}$ which are constant on the B^n_m and $\mathcal{S}_n : \mathcal{M} \to \mathbb{R}$ is

$$\begin{aligned}\mathcal{S}_n(r, \theta, \phi) = \int_\Gamma & \mathrm{d}\rho(x) r(x) \sum_\mu \mathrm{e}^\mu(\theta(x), \phi(x)) Q^\mu_1(g_n(x)) \\ & + \frac{1}{2}\int_{\Gamma\times\Gamma} \mathrm{d}\rho(x)\,\mathrm{d}\rho(y) r(x) r(y) \sum_{\mu,\mu'} \mathrm{e}^\mu(\theta(x), \phi(x)) \\ & \times Q^{\mu\mu'}_2(g_n(x); g_n(y))\, \mathrm{e}^{\mu'}(\theta(y), \phi(y)) - \frac{1}{\beta}\int_\Gamma \mathrm{d}\rho(x) I(r(x)).\end{aligned} \tag{3.5}$$

Defining as usual the free energy $f_N(\beta, \boldsymbol{\xi}) = -(\beta N)^{-1} \log \operatorname{tr} \exp(-\beta H_N(\boldsymbol{\xi}))$, one is drawn to replace g_n by the identity map in (3.5) and conclude the following.

Theorem 2. P_ρ-almost surely

$$\lim_{N\to\infty} f_N(\beta, \boldsymbol{\xi}) = f(\beta) \equiv -\sup_{(r,\theta,\phi)\in\mathcal{M}} \mathcal{S}(r, \theta, \phi)$$

where

$$\begin{aligned}\mathcal{S}(r, \theta, \phi) = \int_\Gamma & \mathrm{d}\rho(x) r(x) \sum_\mu \mathrm{e}^\mu(\theta(x), \phi(x)) Q^\mu_1(x) \\ & + \frac{1}{2}\int_{\Gamma\times\Gamma} \mathrm{d}\rho(x)\,\mathrm{d}\rho(y) r(x) r(y) \sum_{\mu,\mu'} \mathrm{e}^\mu(\theta(x), \phi(x)) \\ & \times Q^{\mu\mu'}_2(x; y)\, \mathrm{e}^{\mu'}(\theta(y), \phi(y)) - \frac{1}{\beta}\int_\Gamma \mathrm{d}\rho(x) I(r(x)).\end{aligned} \tag{3.6}$$

This turns out to be the case. For the proof we will need the following lemma about continuity of the family $\mathcal{S}_n$.

Lemma 3. Let $\boldsymbol{m}$ denote the triple (r, θ, ϕ) in $\mathcal{M}$. Then

$$\lim_{n\to\infty} |\mathcal{S}(\boldsymbol{m}) - \mathcal{S}_n(\boldsymbol{m})| = 0$$

convergence being uniform in $\mathcal{M}$.

Proof.

$$\begin{aligned}|\mathcal{S}(\boldsymbol{m}) - \mathcal{S}_n(\boldsymbol{m})| & \\ \leq \int_\Gamma & \mathrm{d}\rho(x) \sum_\mu |Q^\mu_1(g_n(x)) - Q^\mu_1(x)| \\ & + \frac{1}{2}\int_{\Gamma\times\Gamma} \mathrm{d}\rho(x)\,\mathrm{d}\rho(y) \sum_{\mu,\mu'} |Q^{\mu\mu'}_2(g_n(x); g_n(y)) - Q^{\mu\mu'}_2(x; y)|\end{aligned} \tag{3.7}$$

for all $\boldsymbol{m} \in \mathcal{M}$. By assumption ($b$), and the properties of g_n, the integrands in (3.7) converge pointwise to zero as $n \to \infty$, except possibly on a set of measure zero. Now $\int_\Gamma \mathrm{d}\rho(x)|Q_1^\mu(x)|$ and $\int_{\Gamma\times\Gamma} \mathrm{d}\rho(x)\,\mathrm{d}\rho(y)|Q_2^{\mu\mu'}(x, y)|$ are finite by assumption (b); and $\int_\Gamma \mathrm{d}\rho(x)|Q_1^\mu(g_n(x))|$ and $\int_{\Gamma\times\Gamma} \mathrm{d}\rho(x)\,\mathrm{d}\rho(y)|Q_2^{\mu\mu'}(g_n(x); g_n(y))|$ are just Riemann sums approximating them. Hence the integrals in (3.7) are bounded in n and we can use the dominated convergence theorem to conclude that their limit is zero. □

Next we show that the continuum limit of the discrete free energies $f^n(\beta)$ is given by the supremum of $\mathcal{S}$ over $\mathcal{M}$.

Proposition 4.

$$\lim_{n\to\infty} f^n(\beta) = f(\beta).$$

Proof. $\mathcal{M}_n$ is compact and $\mathcal{S}_n$ clearly continuous, and so there exists $\bar{\boldsymbol{m}}_n \in \mathcal{M}_n$ such that $f^n(\beta) = -\mathcal{S}_n(\bar{\boldsymbol{m}}_n)$. Now

$$\begin{aligned} f^n(\beta) &= -\mathcal{S}(\bar{\boldsymbol{m}}_n) + (\mathcal{S}(\bar{\boldsymbol{m}}_n) - \mathcal{S}_n(\bar{\boldsymbol{m}}_n)) \\ &\geqslant f(\beta) + (\mathcal{S}(\bar{\boldsymbol{m}}_n) - \mathcal{S}_n(\bar{\boldsymbol{m}}_n)). \end{aligned}$$

Thus by lemma 3,

$$\liminf_{n\to\infty} f^n(\beta) \geqslant f(\beta).$$

If we can just show that $f(\beta) \geqslant \limsup_{n\to\infty} f^n(\beta)$ then we are done.

By definition of the supremum, there exist elements $\{\boldsymbol{m}_\alpha : \alpha = 1, 2, \ldots\}$ of $\mathcal{M}$ such that $f(\beta) = -\lim_{\alpha\to\infty} \mathcal{S}(\boldsymbol{m}_\alpha)$. Clearly $\bigcup_n \mathcal{M}^n$ is dense in $\mathcal{M}$ so for each $\boldsymbol{m}_\alpha$ we can find a sequence $\{\boldsymbol{m}_{\alpha,n}\}$ with each $\boldsymbol{m}_{\alpha,n}$ in $\mathcal{M}^n$ such that $\lim_{n\to\infty} \boldsymbol{m}_{\alpha,n} = \boldsymbol{m}_\alpha$. Now

$$\begin{aligned} -\mathcal{S}(\boldsymbol{m}_{\alpha,n}) &= -\mathcal{S}_n(\boldsymbol{m}_{\alpha,n}) + (\mathcal{S}_n(\boldsymbol{m}_{\alpha,n}) - \mathcal{S}(\boldsymbol{m}_{\alpha,n})) \\ &\geqslant f^n(\beta) + (\mathcal{S}_n(\boldsymbol{m}_{\alpha,n}) - \mathcal{S}(\boldsymbol{m}_{\alpha,n})) \end{aligned}$$

so taking $\limsup_{n\to\infty}$ and then $\lim_{\alpha\to\infty}$

$$f(\beta) \geqslant \limsup_{n\to\infty} f^n(\beta)$$

as required. □

It just remains to show that the approximation of the finite volume free energy by its discrete versions is exact in the thermodynamic limit.

Proof of theorem 2. By Bogolubov's inequality

$$|f_N^n(\beta, \boldsymbol{\xi}) - f_N(\beta, \boldsymbol{\xi})| \leqslant R_N^n(\boldsymbol{\xi}) \equiv \frac{1}{N} \|H_N^n(\boldsymbol{\xi}) - H_N(\boldsymbol{\xi})\|. \tag{3.8}$$

Let us examine the RHS of (3.8) first. Let $P_N(\boldsymbol{\xi}; \cdot)$ be the empirical measure $N^{-1}\sum_{i=1}^N \delta_{\xi_i}(\cdot)$ on Γ, i.e. the distribution of the first N elements of a given $\boldsymbol{\xi}$. Then

$$\begin{aligned} R_N^n(\boldsymbol{\xi}) \leqslant \sum_\mu \int_\Gamma & \mathrm{d}P_N(\boldsymbol{\xi}, x)|Q_1^\mu(x) - Q_1^\mu(g_n(x))| \\ & + \frac{1}{2}\sum_{\mu,\mu'} \int_{\Gamma\times\Gamma} \mathrm{d}P_N(\boldsymbol{\xi}, x)\,\mathrm{d}P_N(\boldsymbol{\xi}, y)|Q_2^{\mu\mu'}(x, y) - Q_2^{\mu\mu'}(g_n(x), g_n(y))|. \end{aligned} \tag{3.9}$$

By the ergodic theorem

$$\operatorname*{weak\,lim}_{N\to\infty} P_N(\boldsymbol{\xi}) = \rho. \tag{3.10}$$

P_ρ-almost surely, so since the integrands in (3.9) are piecewise continuous and ρ continuous we can take the limit as $N\to\infty$ and replace $P_N(\boldsymbol{\xi})$ with ρ in the RHS of (3.9). Then taking the limit $n\to\infty$ we get zero, as in proposition 4. To summarise

$$\lim_{n\to\infty}\lim_{N\to\infty} R_N^n(\boldsymbol{\xi}) = 0 \qquad P_\rho\text{-almost surely.}$$

By (3.8)

$$f_N^n(\beta,\boldsymbol{\xi}) - R_N^n(\boldsymbol{\xi}) \leqslant f_N(\beta,\boldsymbol{\xi}) \leqslant f_N^n(\beta,\boldsymbol{\xi}) + R_N^n(\boldsymbol{\xi}). \tag{3.11}$$

Now for any $\boldsymbol{\xi}$ for which (3.10) is obeyed, then clearly

$$\operatorname*{weak\,lim}_{N\to\infty} P_N(\boldsymbol{\xi}) g_n^{-1} = \rho_n$$

and we can apply theorem 1 to the second inequality of (3.11):

$$\limsup_{N\to\infty} f_N(\beta,\boldsymbol{\xi}) \leqslant f_n(\beta) + \lim_{N\to\infty} R_n^N$$

and so

$$\limsup_{N\to\infty} f_N(\beta,\boldsymbol{\xi}) \leqslant f(\beta) \qquad P_\rho\text{-almost surely.}$$

Similarly we conclude from the lower bound for $f_N(\beta,\boldsymbol{\xi})$ that

$$\liminf_{N\to\infty} f_N(\beta,\boldsymbol{\xi}) \geqslant f(\beta) \qquad P_\rho\text{-almost surely}$$

and this completes the proof. □

In the case of a discrete probability distribution, the supremum in theorem 1 is trivially attained. For continuous distributions this is not immediately clear, so we prove the following.

Proposition 5. $\sup_{\boldsymbol{m}\in\mathcal{M}} \mathcal{S}(\boldsymbol{m})$ is attained.

Proof. By definition there exists a sequence $\boldsymbol{m}_n$ in $\mathcal{M}$ such that $\lim_{n\to\infty}\mathcal{S}(\boldsymbol{m}_n) = \sup_{\boldsymbol{m}}\mathcal{S}(\boldsymbol{m})$. Let $\tilde{\mathcal{M}}$ denote the set of functions from Γ to the unit ball B_1 of $\mathbb{R}^3$. We can identify $\mathcal{M}$ with the ρ-measurable functions on $\tilde{\mathcal{M}}$. The topology of pointwise convergence in $\tilde{\mathcal{M}}$ is just the product topology on $(B_1)^{\times\Gamma}$, which is compact by Tychonoff's theorem (see e.g. [21]). Thus, some subsequence of the $\boldsymbol{m}_n$ converges pointwise to some $\tilde{\boldsymbol{m}}$ in $\tilde{\mathcal{M}}$. Since ρ has an expectation and the $\boldsymbol{m}_n$ are bounded, then by the dominated convergence theorem, $\tilde{\boldsymbol{m}}$ lies in $\mathcal{M}$. It is also clear, by use of the dominated convergence theorem, that $\mathcal{S}$ is continuous in the topology of pointwise convergence in $\mathcal{M}$. Hence

$$\lim_{n\to\infty}\mathcal{S}(\boldsymbol{m}_n) = \mathcal{S}(\tilde{\boldsymbol{m}})$$

along the subsequence, and the supremum is obtained. □

4. The mean-field Heisenberg random-field model

We first analyse the thermodynamics of a model with a discrete probability distribution. Let

$$-H_N(\boldsymbol{\xi}) = h\sum_{i=1}^{N}\xi_i\sigma_i^z + \frac{1}{2N}\sum_{i,j=1}^{N}\sigma_i\cdot\sigma_j$$

where each ξ_i takes the values ± 1 with probability $\frac{1}{2}$, and $h \geq 0$. Thus in the terminology of § 2, $\Gamma = \mathbb{R}$, $\rho = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$, $Q_1^\mu(\xi) = h\xi\delta^{\mu z}$ and $Q_2^{\mu\mu'}(\xi, \eta) = \delta^{\mu\mu'}$. For this model, theorem 1 yields that the negative free energy $-f(\beta, h)$ is the supremum of the functional

$$\begin{aligned}\mathscr{S}(r_1, r_2, \theta_1, \theta_2, \phi_1, \phi_2)\\ &= h\left(\frac{r_1}{2}\cos\theta_1 + \frac{r_2}{2}\cos\theta_2\right)\\ &+\frac{1}{2}\left(\frac{r_1^2+r_2^2}{4} - \frac{r_1r_2}{2}(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2))\right)\\ &-\frac{1}{2\beta}(I(r_1)+I(r_2))\end{aligned} \tag{4.1}$$

where we have made the change $\theta_2 \to \pi - \theta_2$ in the variables of (2.2).

Proposition 6. Let $\beta_h = h^{-1}\tanh^{-1}h$ when $h < 1$ and $+\infty$ otherwise, and for $\beta \geq \beta_h$ let r_β be the positive solution of $r = \tanh(\beta r)$. Then for $\beta \leq \beta_h$,

$$f(\beta, h) = -\frac{1}{\beta}\log(2\cosh(\beta h))$$

while for $\beta \geq \beta_h$

$$f(\beta, h) = \tfrac{1}{2}(r_\beta^2 - h^2) - \frac{1}{\beta}\log(2\cosh(\beta r_\beta)).$$

Proof. Clearly, the $\phi_{1,2}$ supremum occurs when $\cos(\phi_1 - \phi_2) = 1$. The Euler–Lagrange equations for the remaining variables are

$$h\cos\theta_1 + \tfrac{1}{2}(r_1 - r_2\cos(\theta_1+\theta_2)) - \frac{1}{\beta}I'(r_1) = 0 \tag{R1}$$

$$h\cos\theta_2 + \tfrac{1}{2}(r_2 - r_1\cos(\theta_1+\theta_2)) - \frac{1}{\beta}I'(r_2) = 0 \tag{R2}$$

$$-h\frac{r_1}{2}\sin\theta_1 + \frac{r_1r_2}{4}\sin(\theta_1+\theta_2) = 0 \tag{θ1}$$

$$-h\frac{r_2}{2}\sin\theta_2 + \frac{r_1r_2}{4}\sin(\theta_1+\theta_2) = 0. \tag{θ2}$$

If $r_1 = r_2 = 0$ then $\mathscr{S} = \beta^{-1}\log 2$. If $r_1 = 0$, $r_2 \neq 0$ (the case $r_2 = 0$, $r_1 \neq 0$ is identical) then from (4.1) we see that for a supremum $\cos\theta_2 = 1$; (R1) dictates that $2h = r_2$ and (R2) leads to a value $-h^2/2 + (2\beta)^{-1}\log(2\cosh 2\beta h)$ of $\mathscr{S}$. This is less than $\beta^{-1}\log(2\cosh\beta h)$, a value for $\mathscr{S}$ which, as we shall see, is attained elsewhere. We now comment that since $I'(r) \to \infty$ as $r \to 1$, the upper boundaries $r_1, r_2 = 1$ can never be maximisers. Hence stationary values of $\mathscr{S}$ with $r_1, r_2 \in (0, 1)$ are, provided they exceed the values already identified, the only candidates for suprema.

It remains to consider $r_1, r_2 \neq 0$. $(\theta 1)$ and $(\theta 2)$ yield

$$h \sin \theta_1 = \frac{r_2}{2} \sin(\theta_1 + \theta_2) \qquad h \sin \theta_2 = \frac{r_1}{2} \sin(\theta_1 + \theta_2). \tag{4.2}$$

Thus $\sin \theta_1, \sin \theta_2$ and $\sin(\theta_1 + \theta_2)$ are either all non-zero or all zero. In the former case, elimination of θ_1, θ_2 yields that $r_1, r_2 = r_\beta$. Clearly this requires $\beta > 1$. In fact, from $(\theta 1)$ and $(\theta 2)$ we see that $\theta_1 = \theta_2$ so that $h = r_\beta \cos \theta_1$. Thus for this solution to exist we require that $r_\beta > h$, or equivalently, $\beta > \beta_h$. Writing $I(r) = r \tanh^{-1} r - \log(2 \cosh \tanh^{-1} r)$, we find that the corresponding value of $\mathscr{S}$ for this low-temperature solution is

$$\mathscr{S} = \frac{h^2 - r_\beta^2}{2} + \frac{1}{\beta} \log(2 \cosh(\beta r_\beta)). \tag{4.3}$$

We now examine the case $\sin \theta_1 = \sin \theta_2 = \sin(\theta_1 + \theta_2) = 0$. $\cos \theta_1, \cos \theta_2$ take the values ± 1 and $\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2$. Clearly

$$\mathscr{S}(r_1, r_2, 0, 0, \phi, \phi) \geq \mathscr{S}(r_1, r_2, \pi, \pi, \phi, \phi)$$

so we need only consider the two cases $\cos \theta_1 = \cos \theta_2 = 1$ and $\cos \theta_1 = -\cos \theta_2 = 1$. The Euler–Lagrange equations become

$$h \cos \theta_1 + \cos \theta_1 \left(\cos \theta_1 \frac{r_1}{2} - \cos \theta_2 \frac{r_2}{2} \right) - \frac{1}{\beta} I'(r_1) = 0 \tag{R1*}$$

$$h \cos \theta_2 + \cos \theta_2 \left(\cos \theta_2 \frac{r_2}{2} - \cos \theta_1 \frac{r_1}{2} \right) - \frac{1}{\beta} I'(r_2) = 0. \tag{R2*}$$

We shall see that in general there exist symmetric ($r_1 = r_2$) and non-symmetric ($r_1 \neq r_2$) solutions of (R1*) and (R2*). However, there is never any exchange of stability between them, for the non-symmetric solution is always unstable WRT angular variations. Specifically, with $\cos \theta_{1,2} = \pm 1$ the Hessian matrix for the problem is block diagonal (i.e. the radial-angular blocks are zero) with angular block:

$$\begin{pmatrix} -h \cos \theta_1 (r_1/2) + \cos \theta_1 \cos \theta_2 (r_1 r_2/4) & \cos \theta_1 \cos \theta_2 (r_1 r_2/4) \\ \cos \theta_1 \cos \theta_2 (r_1 r_2/4) & -h \cos \theta_2 (r_2/2) + \cos \theta_1 \cos \theta_2 (r_1 r_2/4) \end{pmatrix}.$$

A necessary condition for stability is that the determinant

$$D = h \cos \theta_1 \cos \theta_2 \frac{r_1 r_2}{8} (2h - (r_1 \cos \theta_1 + r_2 \cos \theta_2)) \tag{4.4}$$

is positive. First we consider $\cos \theta_2 = \cos \theta_2 = 1$. We note immediately that there is only one symmetric solution of (R1*) and (R2*), namely $r_1 = r_2 = \tanh(\beta h)$. For this high-temperature solution $\mathscr{S}$ takes the value $\beta^{-1} \log(2 \cosh(\beta h))$. By adding r_1 to (R2*) and r_2 to (R1*) we see that

$$\tfrac{1}{2}(r_1 + r_2) - h = r_1 - \frac{1}{\beta} \tanh^{-1} r_1 = r_2 - \frac{1}{\beta} \tanh^{-1} r_2. \tag{4.5}$$

When $\beta < 1$, $r \to r - \beta^{-1} \tanh^{-1} r$ is strictly monotonic, and hence r_1 and r_2 are equal, and so equal to $\tanh \beta h$. If $\beta > 1$ then we have the possibility that solutions of (R1*),

$(R2^*)$ exist with r_1 and r_2 distinct. By considering the graph of the function $r \to r - \beta^{-1}\tanh^{-1} r$, we see that this can only occur if $r_1 - \beta^{-1}\tanh^{-1} r_1$ and $r_2 - \beta^{-1}\tanh^{-1} r_2$ are positive and so, by (4.5), $2h - (r_1 + r_2)$ is negative. But this contradicts the positivity of D. Furthermore, we see that when $\beta > \beta_h$, $h - \tanh(\beta h)$ is negative and so the solution $r_{1,2} = \tanh(\beta h)$ is unstable. When $\cos\theta_1 = -\cos\theta_2 = 1$, then in a similar manner to the previous case

$$(r_1 - r_2)/2 - h = r_1 - \frac{1}{\beta}\tanh^{-1} r_1 = \frac{1}{\beta}\tanh^{-1} r_2 - r_2. \tag{4.6}$$

Hence for $\beta < 1$, (R1*) and (R2*) have no solution. For $\beta > 1$ observe that for stability the determinant requires that

$$2h - (r_1 - r_2) < 0. \tag{4.7}$$

Considering again the graph of $r \to r - \beta^{-1}\tanh^{-1} r$ we see that $r_1 < r_\beta < r_2$. Substituting in (4.7) we get $h < 0$, a contradiction. Thus, to conclude, $r_{1,2} = \tanh(\beta h)$ is the maximiser for $\beta \leqslant \beta_h$, while $r_{1,2} = r_\beta$ is the maximiser for $\beta > \beta_h$. □

Proposition 6 shows that the free energy is continuous across a second-order phase transition at the critical temperature $\beta = \beta_h$. Indeed the techniques of theorem 1 can be extended to calculate the gradients of the free energy (see [19]): we may also calculate expectation values. The thermodynamic limit of the magnetisation density $N^{-1}((S_N^x)^2 + (S_N^y)^2)^{1/2}$ orthogonal to the z direction is given by the expression $(r_1 \sin\theta_1 + r_2 \sin\theta_2)/2$, evaluated at the maximiser of $\mathscr{S}$. For β approaching β_h from above this turns out to be of the form $a(\beta - \beta_h)^{1/2}$ (for some constant a) while it is zero for $\beta < \beta_h$, thus demonstrating the expected square-root singularity.

Finally, we note that using the methods of this paper we can show that the Ising model with transverse random field has the same phase diagram as the present model. However, we contrast with the classical Ising model with random field [22]: in this latter model there is a line of *first*-order transitions in the phase diagram. We will return to this point in the conclusion. The phase diagrams of the various models are given in figures 1 and 2. Here $T_h = \beta_h^{-1}$.

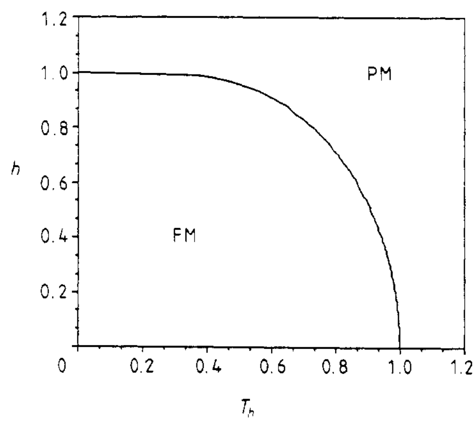

Figure 1. Phase diagram for the quantum random field Heisenberg model of § 4. PM and FM denote the paramagnetic and ferromagnetic phases respectively. The phase boundary is second order.

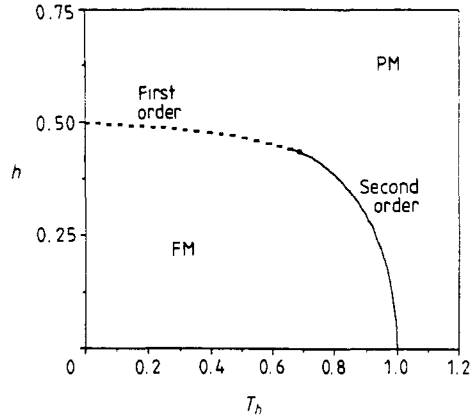

Figure 2. Phase diagram for the classical random field Ising model of [22]. As in figure 1, PM and FM denote the paramagnetic and ferromagnetic phases.

5. A mean-field Heisenberg spin glass with random directions

Our second application of the general theory is the random spin direction model and uses the continuum distribution theory of § 3. To each site we attach two random unit vectors in $\mathbb{R}^3$ whose directions are independent and uniformly distributed. By contraction with the vector of spin components, each vector determines a spin operator. The Hamiltonian is a sum of products of these overall pairs of sites, and can be viewed as a possible quantum generalisation of van Hemmen's classical spin-glass model [7].

In the notation of § 3, $\Gamma=(\mathbb{R}^3)^2$, and ρ is the uniform distribution over $\tilde{\Gamma}=S^3\times S^3\subset\Gamma$. Thus setting $\rho=\nu\otimes\nu$ where ν is the uniform distribution on S^3, we set $Q_1\equiv 0$. We write each $x\in\tilde{\Gamma}$ as a pair of unit vectors $(\mathbf{\Omega}_1(x), \mathbf{\Omega}_2(x))$ and define $Q_2:\tilde{\Gamma}\rightarrow M_3$ by

$$Q_2^{\mu\mu'}(x,y)=(\Omega_1^{\mu}(x)\Omega_2^{\mu'}(y)+\Omega_2^{\mu}(x)\Omega_1^{\mu'}(y)).$$

Thus, writing the vector $\boldsymbol{e}(\theta(x),\phi(x))$ as $\mathbf{\Omega}(x)$, theorem 2 yields that

$$f(\beta)=-\sup_{r\in\mathcal{R},\Omega\in\mathcal{C}}\left\{\mathcal{H}(r,\mathbf{\Omega})-\frac{1}{\beta}\mathcal{I}(r)\right\}$$

where $\mathcal{R}$ is the space of ρ-measurable functions taking values in $[0,1]$, $\mathcal{C}$ is the space of ρ-measurable functions on $\tilde{\Gamma}$ taking values in S^3,

$$\mathcal{H}(r,\mathbf{\Omega})=\frac{1}{2}\int_{\tilde{\Gamma}\times\tilde{\Gamma}}\mathrm{d}\rho(x)\,\mathrm{d}\rho(y)r(x)r(y)$$
$$\times(\mathbf{\Omega}(x)\cdot\mathbf{\Omega}_1(x)\mathbf{\Omega}_2(y)\cdot\mathbf{\Omega}(y)+\mathbf{\Omega}(x)\cdot\mathbf{\Omega}_2(x)\mathbf{\Omega}_1(y)\cdot\mathbf{\Omega}(y))$$

and

$$\mathcal{I}(r)=\int_{\tilde{\Gamma}}\mathrm{d}\rho(x)I(r(x)). \tag{5.1}$$

The solution to the variational problem is the following.

Proposition 7.

$$f(\beta)=-\tfrac{1}{2}\alpha^2+\frac{1}{\beta}\int_{\tilde{\Gamma}}\mathrm{d}\rho(x)c'(\alpha\beta(1+\omega(x))^{1/2}) \tag{5.2}$$

where $\omega(x)=\mathbf{\Omega}_1(x)\cdot\mathbf{\Omega}_2(x)$ and $\alpha=0$ if $\beta\leqslant 1/c''(0)=1$, while if $\beta>1/c''(0)$ then α is the unique positive solution of

$$\alpha=\int_{\tilde{\Gamma}}\mathrm{d}\rho(x)(1+\omega(x))^{1/2}c'(\alpha\beta(1+\omega(x))^{1/2}) \tag{5.3}$$

and

$$c(y)=\sup_{x\in[0,1]}\{xy-I(x)\}=\log(2\cosh y)$$

is the Legendre transform of I.

Proof.
Angular supremum. We show that $\sup_{\mathbf{\Omega}\in\mathcal{C}}\mathcal{H}(r,\mathbf{\Omega})$ is attained when

$$\mathbf{\Omega}(x)=\frac{\mathbf{\Omega}_1(x)+\mathbf{\Omega}_2(x)}{(2(1+\omega(x)))^{1/2}} \tag{5.4}$$

for $\omega(x)\neq-1$, with $\Omega(x)$ arbitrary on the ρ-measure zero set for which $\omega(x)=-1$.

The supremum is attained, and since $\mathcal{C}$ has no boundary and $\mathcal{H}(r,\cdot)$ is continuous, the supremum is attained at a stationary point of $\mathcal{H}(r,\cdot)$. Let v be an $L^\infty(\rho)$ function on from $\tilde{\Gamma}$ to $\mathbb{R}^3$ and for $\mathbf{\Omega}\in\mathcal{C}$, $t\in\mathbb{R}$ define the perturbation $\mathbf{\Omega}_t$ of $\mathbf{\Omega}$ by

$$\mathbf{\Omega}_t(x)=\frac{\mathbf{\Omega}(x)+tv(x)}{(1+t^2+2t\mathbf{\Omega}(x)\cdot v(x))^{1/2}} \tag{5.5}$$

for t sufficiently small for (5.5) to exist everywhere:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(r,\mathbf{\Omega}_t)|_{t=0}=\int_{\tilde{\Gamma}}\mathrm{d}\rho(x)r(x)(J_2\mathbf{\Omega}_1(x)+J_1\mathbf{\Omega}_2(x))\cdot(v(x)-v(x)\cdot\mathbf{\Omega}(x)\mathbf{\Omega}(x))$$

where

$$J_{1,2}=\int_{\tilde{\Gamma}}\mathrm{d}\rho(x)r(x)\mathbf{\Omega}(x)\cdot\mathbf{\Omega}_{1,2}(x). \tag{5.6}$$

The range of $v(x)\to v(x)-(v(x)\cdot\mathbf{\Omega}(x))\mathbf{\Omega}(x)$ is simply the subspace of $\mathbb{R}^3$ orthogonal to $\mathbf{\Omega}(x)$, so for stationairty $\mathbf{\Omega}(x)$ must be parallel to $J_2\mathbf{\Omega}_1(x)+J_1\mathbf{\Omega}_2(x)$ for all x in supp r. We restrict J_1 and J_2 to be both either positive or negative, since otherwise $\mathcal{H}(r,\mathbf{\Omega})=J_1J_2\leqslant 0$. Writing

$$\mathbf{\Omega}(x)=\frac{J_2\mathbf{\Omega}_1(x)+J_1\mathbf{\Omega}_2(x)}{(J_1^2+J_2^2+2J_1J_2\omega(x))^{1/2}} \tag{5.7}$$

when the latter is defined (with $\mathbf{\Omega}(x)$ arbitrary on the set of ρ-measure zero where it is not) and inserting into (5.6):

$$J_1=J_1L+J_2K \qquad \text{and} \qquad J_2=J_2L+J_1K \tag{5.8}$$

where

$$L=\int_{\tilde{\Gamma}}\mathrm{d}\rho(x)\frac{r(x)\omega(x)}{(J_1^2+J_2^2+2J_1J_2\omega(x))^{1/2}}$$

$$K=\int_{\tilde{\Gamma}}\mathrm{d}\rho(x)\frac{r(x)}{(J_1^2+J_2^2+2J_1J_2\omega(x))^{1/2}}.$$

Clearly, K, and hence $1-L$, is non-zero, so from (5.8), $J_1^2=J_2^2$. Hence $J_1=J_2$ and $\mathscr{H}(r,\mathbf{\Omega})$ is positive. Substituting in (5.1), J_1 and J_2 drop out, yielding

$$\mathscr{H}(r,\mathbf{\Omega})-\frac{1}{\beta}\mathscr{I}(r)=\tilde{\mathscr{F}}(r)\equiv\left(\int_{\tilde{\Gamma}}\mathrm{d}\rho(x)r(x)(1+\omega(x))^{1/2}\right)^2-\frac{1}{\beta}\mathscr{I}(r).$$

Radial supremum. We will find the stationary points of $\tilde{\mathscr{F}}$ and eliminate the possibility of non-stationary suprema on the boundaries $r(x)=0,1$. For $f\in\mathscr{L}^\infty(\tilde{\Gamma},\rho)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathscr{F}}(r+tf)\bigg|_{t=0}\\ &=\int_{\tilde{\Gamma}}\mathrm{d}\rho(x)f(x)(1+\omega(x))^{1/2}\int_{\tilde{\Gamma}}\mathrm{d}\rho(y)r(y)(1+\omega(y))^{1/2}\\ &\quad-\frac{1}{\beta}\int_{\tilde{\Gamma}}\mathrm{d}\rho(x)f(x)I'(r(x)).\end{aligned}$$

Already we see that since $I'(r)=\tanh^{-1}(r)$, $I'(r)\to\infty$ as $r\to1$ so that the supremum r^* must be bounded away from 1. (A rigorous justification of this argument is provided in [19]). For the lower boundary, let $B=\{x:r^*(x)=0\}$ and choose $f=\chi_B$. Then since $I'(0)=0$,

$$\int_B\mathrm{d}\rho(x)(1+\omega(x))^{1/2}\int_{B'}\mathrm{d}\rho(y)r^*(y)(1+\omega(y))^{1/2}\geq0$$

so that $\rho(B)$ is 0 or 1 for r^* to be a maximiser. Note at this point that $I'=(c')^{-1}$ the condition on I can be reexpressed as

$$c'(\infty)=1\qquad c'(0)=0. \tag{5.9}$$

(For general properties of the Legendre transform see [23].)

The local stability of the stationary point $r^*=0$ is obtained from the second derivatives:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{\mathscr{F}}(tf)\bigg|_{t=0}=\left(\int_{\tilde{\Gamma}}\mathrm{d}\rho(x)f(x)(1+\omega(x))^{1/2}\right)^2-\frac{1}{\beta}\int_{\tilde{\Gamma}}\mathrm{d}\rho(x)f^2(x)I''(0).$$

Let A be the integral operator with kernel $A(x,y)=(1+\omega(x))^{1/2}(1+\omega(y))^{1/2}$ and note that

$$I''(0)=\frac{1}{c''((c')^{-1}(0))}=\frac{1}{c''(0)}.$$

Then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{\mathscr{F}}(tf)\bigg|_{t=0}=\left\langle f,\left(A-\frac{1}{\beta c''(0)}1\right)f\right\rangle.$$

It is easily shown that $\|A\|=1$, so that $r^*=0$ is stable if $\beta<(c''(0))^{-1}$. Conversely, taking $f(x)=(1+\omega(x))^{1/2}$ we see that $r^*=0$ is unstable if $\beta>(c''(0))^{-1}$. We now show that in the latter case there exists only one other stationary point, which is hence the maximiser. In the former case, there is no other stationary point, so $r^*=0$ is the maximiser.

Letting $\alpha = \int_{\tilde{\Gamma}} d\rho(x) r^*(x)(1+\omega(x))^{1/2}$ then the condition for $(d/dt)\tilde{\mathscr{F}}(r^* + tf)|_{t=0}$ to be zero for arbitrary f is that

$$0 = F(\alpha) \equiv \alpha - \int_{\tilde{\Gamma}} d\rho(x)(1+\omega(x))^{1/2} c'(\alpha\beta(1+\omega(x))^{1/2}). \tag{5.10}$$

Clearly $\alpha = 0$ is a solution of (5.10). Now,

$$F'(\alpha) = 1 - \beta \int_{\tilde{\Gamma}} d\rho(x)(1+\omega(x)) c''(\alpha\beta(1+\omega(x))^{1/2}).$$

Since $c''(x) = \mathrm{sech}^2(x)$ is decreasing, $\alpha \to F'(\alpha)$ is strictly increasing. Thus $F(\alpha) = 0$ will have a unique strictly positive solution if and only if $0 > F'(0) = 1 - \beta c''(0)$, as required. The explicit form (5.2) follows straightforwardly. □

Comparison with a classical model. We emphasise that the result obtained is not just the free energy of the corresponding model of classical Heisenberg spins. However, the thermodynamics are similar, although the details are different.

Let $s_i : i = 1, \ldots, N$ in S^3 and define the random classical Hamiltonian

$$H_{N,\mathrm{cl}}(\boldsymbol{\xi}, s) = -\frac{1}{2N} \sum_{i,j=1}^{N} s_i \cdot Q_2(\xi_i ; \xi_j) \cdot s_j$$

with Q_2 and $\boldsymbol{\xi}$ as before. Then one sees easily from [11] that the free energy is almost surely given by an expression of the form (5.1), but with I replaced by I_{cl}, the rate function occurring for the large deviation principle for the distribution of the components of classical spins: I_{cl} is the Legendre transform of c_{cl}, the cumulant generating functional for a component of the sum of independent identically distributed spins on the sphere:

$$c_{\mathrm{cl}}(|y|) = \log \int_{S^3} \frac{d\Omega}{4\pi} e^{y \cdot \Omega} = \frac{\sinh|y|}{|y|}.$$

One easily verifies conditions (5.9) for c_{cl}. Furthermore c''_{cl} is decreasing and $c''_{\mathrm{cl}}(0) = \frac{1}{3}$. Thus proposition 7 carries over using c_{cl} instead of c, and the critical temperature is $\frac{1}{3}$.

6. Conclusions

We have provided a method to treat the thermodynamics of site-random quantum mean-field systems. The methods have enabled us to find the phase structure in some models. In § 5, we saw that the results for the quantum spin glass were qualitatively the same as those for the corresponding classical Heisenberg model. In fact, the same statement can be made for the random-field model of § 4: one needs only to replace the function c by c_{cl} throughout. We noted in § 4 that the phase diagram of the classical Ising random-field model *is* qualitatively different: this latter model has a line of first-order transitions. We therefore conclude that the first-order transition does not appear in the quantum and classical Heisenberg random-field models because of the extra degrees of freedom in these models: these relieve the frustration present in the classical Ising random-field model. The relief is not *per se* a quantum phenomenon.

In outlook we reiterate that our methods can be used to examine the thermodynamics of arbitrary quantum mean-field random-site models with finitely many random variables per site. As in the classical case [11], the methods also carry over to any multispin (i.e. polynomial) interaction.

Acknowledgment

We thank V A Zagrebnov for an invitation to speak about our results at JINR, Dubna, USSR, and for his constructive comments on the relationship between classical and quantum models.

References

[1] Binder K and Young A P 1986 *Rev. Mod. Phys.* **58** 801
Mézard M, Parisi G and Virasoro M A 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
[2] Stinchcombe R B 1983 *Phase Transitions and Critical Phenomena* vol 7 ed D Domb and J L Lebowitz (New York: Academic) p 151
[3] Imbrie J Z 1986 *Physica* **140A** 291 (*Proc. STATPHYS 16, Boston*)
[4] Fröhlich J 1986 *Critical Phenomena, Random Systems, and Gauge Theories* part II ed K Osterwalder and R Stora (Amsterdam: North-Holland) p 727
[5] McCoy B M and Wu T T 1968 *Phys. Rev.* **176** 631
[6] Sherrington D and Kirkpartrick S 1975 *Phys. Rev. Lett.* **35** 1792
[7] van Hemmen J L 1982 *Phys. Rev. Lett.* **49** 409
van Hemmen J L, van Enter A C D and Canisius J 1983 *Z. Phys.* B **50** 311
[8] Provost J P and Vallée G 1983 *Phys. Rev. Lett.* **50** 589
[9] Parisi G 1980 *J. Phys. A: Math. Gen.* **13** 1101
[10] Grensing D and Kühn R 1986 *J. Phys. A: Math. Gen.* **19** L1153; 1987 *J. Physique* **48** 713
[11] van Hemmen J L, Grensing D, Huber A and Kühn R 1986 *Z. Phys.* B **65** 53
[12] Berezin F A 1972 *Math. USSR Izv.* **6** 117
[13]. Lieb E H 1973 *Commun. Math. Phys.* **31** 327
[14] Simon B 1980 *Commun. Math. Phys.* **71** 247
[15] Hepp K and Lieb E H 1973 *Phys. Rev.* A **8** 2517
[16] Cegła W, Lewis J T and Raggio G A 1988 *Commun. Math. Phys.* **118** 337
[17] Ellis R S 1985 *Entropy, Large Deviations and Statistical Mechanics* (Berlin: Springer)
[18] Varadhan S R S 1966 *Commun. Pure Appl. Math.* **19** 261
[19] Duffield N G and Pulé J V 1988 *Commun. Math. Phys.* **118** 475
[20] Duffield N G and Pulé J V 1989 *J. Stat. Phys.* **54** 449
[21] Reed M and Simon B 1975 *Functional Analysis* (New York: Academic)
[22] Salinas S R and Wreszinski W F 1985 *J. Stat. Phys.* **41** 299
[23] Rockafellar R T 1970 *Convex Analysis* (Princeton, NJ: Princeton University Press)